

Invariants from classical field theory

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Abstract

We introduce a method that generates invariant functions from perturbative classical field theories depending on external parameters. Applying our methods to several field theories such as abelian BF , Chern-Simons and 2-dimensional Yang-Mills theory, we obtain, respectively, the linking number for embedded submanifolds in compact varieties, the Gauss' and the second Milnor's invariant for links in S^3 , and invariants under area-preserving diffeomorphisms for configurations of immersed planar curves.

1 Introduction

Suppose we have a physical system determined by an action functional S depending on fields as usual, but also on external parameters p belonging to some space P . Fixing the external parameters we can study our field theory from a classical or quantum point of view. A fundamental problem in physics is to understand how the system changes as p varies. In this work we focus our attention on a particular issue arising within this setting. Suppose that fields and external parameters are acted upon by a Lie group G , and that S is invariant under the simultaneous action of G on fields and parameters. How is the group of symmetries G reflected on the states of the system as p varies? The answer to this problem is very different depending on whether we look at it from a quantum or a classical point of view. The quantum situation requires techniques related to anomalies and will not be discussed in this work.

We look at the classical situation from two points of view: we consider exact as well as perturbative solutions of the equations of motion. The exact results admit a simple and clean description given in Section 2. Simplifying slightly, the conclusion is that associated with such a classical system there is a function $S_{os} : P \longrightarrow \mathbb{R}$ whose value at p is obtained by evaluating S on-shell, i.e. $S_{os}(p) = S(\varphi(p), p)$ where $\varphi(p)$ represents a solution of the equations of motion of the action $S(\cdot, p)$. If the correspondence $p \longrightarrow \varphi(p)$ can be constructed in a G -equivariant

fashion, then the function $S_{os} : P \longrightarrow \mathbb{R}$ is invariant under the action of G . We apply this technique to abelian BF theory on compact oriented manifolds, with the external parameters being pairs of non-intersecting embedded submanifolds of the appropriated dimensions. The on-shell action is the linking number of the embedded submanifolds, an invariant under diffeomorphism of the ambient manifold connected to the identity. We write explicitly using local coordinates the on-shell action for this example.

It is seldom possible to find explicitly the solutions of the equations of motion for most interesting Lagrangian systems. In this case an alternative route is to look for perturbative solutions of the equation of motion. Section 4 contains the main result of this work: the proof that a hierarchy of invariant functions on parameter space can be obtained from the computation of the perturbative on-shell action S_{os} of a functional action S invariant under the simultaneous action of a Lie group on fields and external parameters. In order to show this result we develop in Section 3 a fairly explicit model for the study of perturbative solutions of systems of equations (algebraic, differential, integral, etc.) We show that, in the non-degenerate case, the perturbative solution of the system is unique. In the degenerated case, even though uniqueness is lost, our methods guarantee that the space of perturbative solutions is non-empty, and provide an explicit solution under weak assumptions. We apply this technique to solve a couple of general equations arising in the context of Hodge algebras using techniques closely related to homological perturbation theory [18]. In Section 5 we consider an example of interest in low dimensional topology, namely, we applied our methodology to Chern-Simons-Wong action and show that it yields link invariants, in particular, we obtain the Gauss' and the second Milnor's invariants. In Section 6 we discuss how invariants under area preserving diffeomorphisms of \mathbb{R}^2 can be obtained given a generic finite family of immersed curves in the plane, applying our methods to Yang-Mills-Wong action. The Wong term that we add both in the Chern-Simons and Yang-Mills theories may be thought, physically, as the action for conservation of chromo-electric charge, and mathematically, as the action functional for parallel transport in a fiber bundle. In Section 7 we present a brief discussion of open problems and future lines of research.

2 Exact results

Fix spaces F and P , thought as the space of fields and parameters, respectively. Assume that a Lie group K , thought as the gauge group, acts on F . Fix another Lie group G which acts on F and P , together with a map $k : G \longrightarrow K$. Let G act on $F \times P$ via the diagonal action. Suppose we have map $S : F \times P \longrightarrow \mathbb{R}$, thought as the action of a classical field theory, satisfying $S(g\varphi, gp) = S(\varphi, p)$ and $S(k\varphi, p) = S(\varphi, p)$ for $(\varphi, p) \in F \times P$, $g \in G$ and $k \in K$. In addition, assume we have map $\alpha : P \longrightarrow F$ such that $\alpha(gp) = k(g)g\alpha(p)$ for $g \in G$ and $p \in P$. Lemma 1

below explains how one can get a G -invariant function on P from this data.

Lemma 1. The map $S_\alpha : P \longrightarrow \mathbb{R}$ given by $S_\alpha(p) = S(\alpha(p), p)$ is G -invariant.

Indeed for $g \in G$ and $p \in P$ we have that:

$$S_\alpha(gp) = S(\alpha(gp), gp) = S(k(g)g\alpha(p), gp) = S(g\alpha(p), p) = S(\alpha(p), p) = S_\alpha(p).$$

A fundamental question thus arises in this context: how can one obtain such a map α ? We are going to show via examples that it is often possible to find a map α with the required properties by solving the equation of motion, i.e. finding for each $p \in P$ a solution $\varphi(p)$ of the equation

$$\frac{\partial S}{\partial \varphi}(\varphi(p), p) = 0.$$

Thus we are going to show that the so-called on-shell action $S_{os}(p) = S(\varphi(p), p)$ is a G -invariant function on parameter space.

Let us first consider abelian BF gauge theory generalizing a construction of [24]. Let M be a compact oriented manifold of dimension n and fix $1 \leq p \leq n$. The space of fields is

$$\Omega^p(M) \oplus \Omega^{n-p-1}(M)$$

where $\Omega^i(M)$ denotes the space of differential i -forms on M . Let $BE(M, i)$ be the space of bounding embedded i -dimensional submanifolds of M , i.e.

$$BE(M, i) = \{ \gamma \mid \gamma : \Sigma \rightarrow M \text{ bounding embedding, } \Sigma \text{ a compact oriented } i\text{-manifold} \} / \sim .$$

An embedding $\gamma : \Sigma \longrightarrow M$ is bounding if there exists embedding $\delta : \Delta \longrightarrow M$, where Δ is an oriented manifold with boundaries such that $\partial(\Delta) = \Sigma$ and $\delta|_\Sigma = \gamma$. Embeddings $\gamma_1 : \Sigma_1 \longrightarrow M$ and $\gamma_2 : \Sigma_2 \longrightarrow M$ are \sim equivalent if there exists an orientation preserving diffeomorphism $\phi : \Sigma_1 \longrightarrow \Sigma_2$ such that $\gamma_1 = \phi \circ \gamma_2$. The space of parameters is $BE(M, p) \times BE(M, n-p-1)$ and the action functional

$$S : (\Omega^p(M) \oplus \Omega^{n-p-1}(M)) \times BE(M, p) \times BE(M, n-p-1) \longrightarrow \mathbb{R}$$

is a BF theory couple to external parameters given by

$$-S(A_1, A_2, \gamma_1, \gamma_2) = \int_M A_1 \wedge dA_2 + \int_{\Sigma_1} \gamma_1^* A_1 + \int_{\Sigma_2} \gamma_2^* A_2,$$

for $(A_1, A_2, \gamma_1, \gamma_2) \in (\Omega^p(M) \oplus \Omega^{n-p-1}(M)) \times BE(M, p) \times BE(M, n-p-1)$. The action S is invariant under gauge transformations $A_1 \rightarrow A_1 + df_1$, $A_2 \rightarrow A_2 + df_2$, where $f_1 \in \Omega^{p-1}(M)$ and $f_2 \in \Omega^{n-p-2}(M)$. Let $A(M)$ be the infinite dimensional Lie group of automorphisms of M connected to the identity; $A(M)$ acts on forms and embedded submanifolds by pull back and

push forward, respectively. S is manifestly $A(M)$ -invariant since it is metric-independent.

Recall [17] that the Poincaré dual form $P(\gamma) \in \Omega^{n-p}(M)$ of an embedding $\gamma : \Sigma \longrightarrow M$ is uniquely determined, modulo de addition of an exact form, by demanding that

$$\int_{\Sigma} \gamma^* A = \int_M P(\gamma) \wedge A$$

for $A \in \Omega^p(M)$. Poincaré dual forms have the following properties: if $\gamma : \Sigma \longrightarrow M$ is the boundary of $\delta : \Delta \longrightarrow M$, then $d(P(\Delta)) = P(\Sigma)$; if $\phi : M \longrightarrow M$ is a diffeomorphism, then $P(\phi^{-1} \circ \gamma) = \phi^* P(\gamma)$. Poincaré dual forms, among many other things, are useful to compute the linking number $lk(\gamma_1, \gamma_2)$ of embedded bounding submanifolds $\gamma_1 : \Sigma_1 \longrightarrow M$ and $\gamma_2 : \Sigma_2 \longrightarrow M$ of dimension p and $n - p - 1$, respectively, as follows:

$$lk(\gamma_1, \gamma_2) = \int_M P(\Sigma_1) \wedge P(\Delta_2)$$

where $\delta_2 : \Delta_2 \longrightarrow M$ is such that $\partial(\Delta_2) = \Sigma_2$ and $\partial(\delta_2) = \gamma_2$. Using Poincaré dual forms the action S may be written as:

$$-S(A_1, A_2, \gamma_1, \gamma_2) = \int_M A_1 \wedge dA_2 + \int_M P(\Sigma_1) \wedge A_1 + \int_M P(\Sigma_2) \wedge A_2.$$

Varying S with respect to A_1 and A_2 we obtain the equations of motion

$$dA_1 = (-1)^p P(\Sigma_2) \quad \text{and} \quad dA_2 = (-1)^{p(n-p)+1} P(\Sigma_1).$$

Thus the on-shell action $-S_{os}(A_1, A_2, \gamma_1, \gamma_2)$ is given by

$$\int_M P(\Sigma_2) \wedge A_2 = \int_{\Sigma_2} \gamma_2^* A_2 = \int_{\partial(\Delta_2)} \gamma_2^* A_2 = \int_{\Delta_2} \gamma_2^*(dA_2) = \int_M P(\Sigma_1) \wedge P(\Delta_2).$$

We have shown that the on-shell action is given by

$$-S_{os}(\gamma_1, \gamma_2) = \int_M P(\Sigma_1) \wedge P(\Delta_2).$$

From this expression it is clear that the on-shell action S_{os} is an $A(M)$ -invariant function on $BE(M, p) \times BE(M, n - p - 1)$, indeed $S_{os}(\gamma_1, \gamma_2)$ computes the linking number of the embedded submanifolds γ_1, γ_2 . Let us consider the case where $M = \mathbb{R}^n$ and use coordinates (x_1, x_2, \dots, x_n) to write the solution of the equations of motion and the on-shell action. The Poincaré dual form $P(\gamma) = P(\gamma)_\mu dx_\mu$ of an embedded p -manifold $\gamma : \Sigma \longrightarrow \mathbb{R}^n$ is given by

$$P(\gamma)_\mu(x) = \varepsilon_{\mu, \nu} \int_{\Sigma} \gamma^*(dx^\nu) \delta^n(x - \gamma(a)),$$

where $a \in \Sigma$, $\delta^n(x - \gamma(a))$ is the n -dimensional Dirac's delta function centered at $\gamma(a)$, and for $\mu = (\mu_1, \dots, \mu_p)$ we set $dx^\mu = dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$. The solution $A_\gamma = A_{\gamma, \mu} dx^\mu$ of the equations of motion is given by

$$A_{\gamma, \mu}(x) = \int_{\mathbb{R}^n} dy^n \epsilon_{\mu c \nu} \frac{(x - y)^{\mu c}}{|x - y|^n} P(\gamma_2)_\nu(y),$$

where $|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ and $\epsilon_{\mu c \nu} = \epsilon_{\mu_1 \mu_2 \dots \mu_p c \nu_1 \dots \nu_{n-p-1}}$ is the completely antisymmetric symbol in n -dimensions. Using the expression above for both A_1 and A_2 , we obtain explicitly the on-shell action S_{os} , i.e., the linking number of the embeddings γ_1 and γ_2 :

$$S_{os}(\gamma_1, \gamma_2) = \int_{\gamma_1} \int_{\gamma_2} \epsilon_{\mu c \nu} \gamma_1^*(dx^\mu) \gamma_2^*(dx^\nu) \frac{(\gamma_1(a) - \gamma_2(b))^c}{|\gamma_1(a) - \gamma_2(b)|^n}.$$

For $n = 3$, $p = 1$, γ_1 and γ_2 are actually closed curves in \mathbb{R}^3 and the on-shell action $S_{os}(\gamma_1, \gamma_2)$ is the Gauss' linking number. We may also consider the case $n = 1$, $p = 0$, indeed let us consider a slightly generalized action. The space of fields $C^\infty(\mathbb{R})^n$ consists of n -tuples (f_1, \dots, f_n) of piecewise smooth functions on the real line \mathbb{R} . The manifold of parameters $C_n(\mathbb{R}) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \neq x_j \text{ for } i \neq j\}$ is the space of configurations of n distinguishable points on the real line. Configuration space and its compactification are studied in [1, 8]. The action $S : C^\infty(\mathbb{R})^{\times n} \times C_n(\mathbb{R}) \longrightarrow \mathbb{R}$ given by

$$S(f_1, \dots, f_n; x_1, \dots, x_n) = \sum_{i < j} \int_{\mathbb{R}} f_i(x) f_j'(x) dx - \sum_i f_i(x_i)$$

is invariant under the natural action of the group $A(\mathbb{R})$ of orientation preserving diffeomorphisms of \mathbb{R} . The equations of motion are $-\sum_{j < i} f_j' + \sum_{i < j} f_j' - \delta_{x_i} = 0$, where δ_x is the delta function concentrated in x and $1 \leq i \leq n$. Integrating we get $-\sum_{j < i} f_j + \sum_{i < j} f_j = \theta_{x_i}$, where θ_x denotes the Heaviside theta function with jump at x . Thus the equation of motion is $Af = \theta$, where A is the $n \times n$ matrix given $A_{i,j} = sg(j - i)$, $f = (f_1, \dots, f_n)$ and $\theta = (\theta_{x_1}, \dots, \theta_{x_n})$. One can check that $f = B\theta$ where $B_{i,j} = (-1)^{|i-j|}$ and that the on-shell action $S_{os} : C_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is given by

$$S_{os}(x_1, \dots, x_n) = \sum_{i < j, k, s} (-1)^{|i-k|+|j-s|} \theta_{x_s}(x_k).$$

Notice that $S_{os}(x_1, \dots, x_n)$ is indeed an $A(\mathbb{R})$ -invariant function on configuration space as it should according to Lemma 1.

3 Perturbative solutions

Suppose one is interested in finding solutions of an equation of the form $O(\varphi) = \psi$, where V is a vector space, $O : V \longrightarrow V$ is a non-necessarily linear map, ψ is an element of V , and φ is the unknown. We are actually going to work perturbatively, so we may as well start with a map

$O : V \longrightarrow V[[\lambda]]$, thus $O = \sum_{n=0}^{\infty} O_n \lambda^n$ where $O_n : V \longrightarrow V$ is a non-necessarily linear map. Assume that each O_n admits a globally converging Taylor expansion

$$O_n = \sum_{k=1}^{\infty} O_{n,k}(\varphi, \dots, \varphi),$$

where $O_{n,k} : V^{\otimes k} \longrightarrow V$ is a multilinear operator and $O_{n,1} = 0$ for $n \geq 1$. Finding solutions of the equation $\sum_{n=0}^{\infty} O_n \lambda^n = \psi$ is a notoriously difficult problem, and no general answer should be expected. Remarkably, it can be treated perturbatively as follows: performing the substitutions $\varphi \rightarrow \lambda\varphi$ and $\psi \rightarrow \lambda\psi$, the equation $\sum_{n=0}^{\infty} O_n \lambda^n = \psi$ becomes

$$\sum_{n \geq 0, k \geq 1} O_{n,k}(\varphi, \dots, \varphi) \lambda^{n+k-1} = \psi.$$

Making the expansion $\varphi = \sum_{i=0}^{\infty} \varphi_i \lambda^i$ transforms the equation above, a system of non-linear equations, into a system of infinitely many linear equations. Indeed, taking into account the powers of λ we get a zero order equation:

$$O_{0,1}(\varphi_0) = \psi, \tag{1}$$

and for $n \geq 1$ we get higher order equations:

$$\sum_{m,k,i_1,\dots,i_k} O_{m,k}(\varphi_{i_1}, \dots, \varphi_{i_k}) = 0,$$

where the sum runs over non-negative integers m, k, i_1, \dots, i_k such that $n = m + \sum_{s=1}^k i_s + k - 1$. The equation above may be written in the suggestive form

$$O_{0,1}(\varphi_n) = - \sum_{m,k,i_1,\dots,i_k} O_{m,k}(\varphi_{i_1}, \dots, \varphi_{i_k}), \tag{2}$$

where $m \geq 0, k \geq 2$ and $n = m + \sum_{s=1}^k i_s + k - 1$. Thus necessarily integers i_1, \dots, i_k are strictly less than n , and if $O_{0,1}$ is invertible then (2) uniquely determines φ_n in terms of φ_i with $i < n$.

In order to find φ_n explicitly we need several combinatorial notions [5]. A directed graph is a triple $(V, E, (s, t))$ where V and E are finite sets – the set of vertices and edges – and $(s, t) : E \longrightarrow V \times V$ is a map. A path γ in a graph is a sequence of edges e_1, e_2, \dots, e_k such that $t(e_i) = s(e_{i+1})$ for $1 \leq i \leq k - 1$. We say that γ is path from $s(e_1)$ to $t(e_k)$. A rooted tree T is a directed graph with a distinguished vertex r , called the root, such that for each vertex v of T there is a unique path in T from v to r . The valence of a vertex v is $val(v) = |star(v)|$, where $star(v) = \{e \in E \mid t(e) = v\}$. Vertex v is a leave if $val(v) = 0$. A vertex that is not a leave is called internal. The set of internal vertices is denoted by V_I and the set of leaves is denoted by V_L . The root of a tree is an internal vertex, except in the case of the tree \bullet whose

unique vertex is the root. A tree together with a linear order on $star(v)$, for each internal vertex v is called planar. A labelled planar rooted tree is a pair (T, l) where T is a planar rooted tree and $l : V_I(T) \rightarrow \mathbb{N}$ is a map, called the labelling of T . Consider the category whose objects are labelled planar rooted trees. A morphism is a pair (f, g) where $f : V_{T_1} \rightarrow V_{T_2}$ and $g : E_{T_1} \rightarrow E_{T_2}$ are maps such that $(s_2, t_2) \circ g = (f, f) \circ (s_1, t_1)$; moreover we demand the pair (f, g) preserves both the label and the linear ordering on $star(v)$ for each internal vertex v .

Definition 2. For $n \geq 1$ let T_n be the set of isomorphism classes of labelled planar rooted trees T such that: $val(v) \geq 2$ for $v \in V_I$ and $\sum_{v \in V_I} (val(v) + l(v)) = n + |V_I|$.

A labelled planar rooted tree T is uniquely constructed by joining planar subtrees T_1, \dots, T_k , $k \geq 2$, to the root r labelled by l , see Figure 1. If T is so constructed we write $T = (T_1, \dots, T_k)_l$. The following set theoretical identities hold:

$$V_L(T) = \sqcup_{s=1}^k V_L(T_s) \quad \text{and} \quad V_I(T) - \{r\} = \sqcup_{s=1}^k V_I(T_s).$$

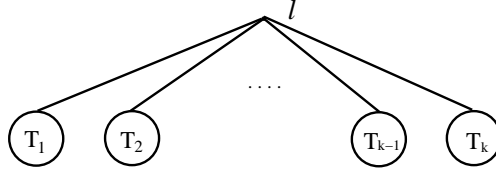


Figure 1: Tree $(T_1, \dots, T_k)_l$.

Lemma 3. Let T_s belong to T_{i_s} for $1 \leq s \leq k$. Then $T = (T_1, \dots, T_k)_l$ belongs to T_n for $n = \sum_{s=1}^k i_s + k + l - 1$.

Proof. The set theoretical identities above imply that

$$\begin{aligned} \sum_{v \in V_I(T)} (val(v) + l(v)) &= \sum_{s, v \in V_I(T_s)} (val(v) + l(v)) + val(r_T) + l = \\ &= \sum_{s=1}^k i_s + \sum_{s=1}^k |V_I(T_s)| + k + l = \sum_{s=1}^k i_s + k + l - 1 + |V_I(T)|. \end{aligned}$$

□

Next definition assumes that the operator $O_{0,1}$ is invertible.

Definition 4. For $T \in T_n$ let $O_T : V^{\otimes |V_L(T)|} \rightarrow V$ be recursively given by

$$O_\bullet = O_{0,1}^{-1} \quad \text{and} \quad O_{(T_1, \dots, T_k)_l} = -O_{0,1}^{-1}(O_{l,k}(O_{T_1}, \dots, O_{T_k})).$$

Proposition 5. The perturbative solution $\varphi = \sum_{n=0}^{\infty} \varphi_n \lambda^n$ of equations (1) and (2) is given by

$$\varphi_n = \sum_{T \in \mathcal{T}_n} O_T(\psi, \dots, \psi). \quad (3)$$

Proof. Let φ_n be given by (3) then $O_{0,1}(\varphi_n) = \sum_{T \in \mathcal{T}_n} O_{0,1}(O_T(\psi, \dots, \psi))$. By Definition 4, Lemma 3, induction and writing $T = (T_1, \dots, T_k)_l$, the previous sum is equal to

$$\sum_{n \geq 0, k \geq 2, T_1 \in \mathcal{T}_{i_1}, \dots, T_k \in \mathcal{T}_{i_k}} -O_{n,k}(O_{T_1}(\psi, \dots, \psi), \dots, O_{T_k}(\psi, \dots, \psi)) = \sum_{n \geq 0, k \geq 2, i_1, \dots, i_k} -O_{n,k}(\varphi_{i_1}, \dots, \varphi_{i_k}).$$

Thus φ_n satisfies the required recursion. \square

Consider the polynomial equation $a_n x^n + \dots + a_2 x^2 + a_1 x = y$ with $a_1 \neq 0$. Instead of looking for an exact expression for x as a function of y we look for a perturbative solution, i.e. a solution $x = \sum_{n=0}^{\infty} x_n \lambda^n$ of the equation $a_n x^n \lambda^{n-1} + \dots + a_2 x^2 \lambda + a_1 x = y$. Let \mathcal{T}_n^0 be the set of isomorphism classes of rooted planar trees with 0 as the label of all internal vertices. Proposition 5 implies that

$$x_n = \sum_{T \in \mathcal{T}_n^0} (-1)^{|V_i(T)|} a_1^{-|V(T)|} a_T y^{|V_i(T)|},$$

where

$$a_T = \prod_{v \in V_i(T)} a_{\text{val}(v)}.$$

Corollary 6. $x = \sum_{n=0}^{\infty} |\mathcal{T}_n^0| \lambda^n$ is the formal solution of $\sum_{n=2}^{\infty} x^n \lambda^{n-1} - x = -1$.

Similarly one can check that:

Corollary 7. $x = \sum_{n=0}^{\infty} |\mathcal{T}_n| \lambda^n$ is the formal solution of $\sum_{n,k \geq 2} x^n \lambda^{n+k-1} - x = -1$.

We say that an operator $O : V \longrightarrow V$ has a right inverse if there exists an operator $P : V \longrightarrow V$ such that $O(P(\varphi)) = \varphi$ for $\varphi \in O(V)$. The proof of Proposition 5 yields the following result.

Proposition 8. Let us assume that $\psi \in O_1(M)$, $O_{0,1}$ posses a right inverse P , and that $\sum_{n \geq 0, k \geq 2, i_1, \dots, i_k} O_{n,k}(\varphi_{i_1}, \dots, \varphi_{i_k}) \in O_1(V)$, where $n = m + \sum_{s=1}^k i_s + k - 1$. A solution of (1) and (2) is given by $\varphi_n = \sum_{T \in \mathcal{T}_n} O_T(\psi, \dots, \psi)$, where $O_{(T_1, \dots, T_k)_l} = -P(O_{l,k}(O_{T_1}, \dots, O_{T_k}))$, $O_{\bullet} = P$.

We refer to the conditions of Proposition 8 as the consistency conditions. To illustrate how Proposition 8 works we solve perturbatively two general equations arising in Hodge algebras [16, 22, 34]. The methods we use to solve these equations resemble the techniques of homological

perturbation theory [18]. Let (A, d, \langle, \rangle) be a Hodge algebra, i.e. (A, d) is a differential graded algebra, $\langle, \rangle: A \otimes A \rightarrow \mathbb{R}$ is a graded symmetric non-degenerated bilinear form, and A admits a Hodge decomposition. The adjoint d^* of d is such that $\langle da, b \rangle = \langle a, d^*b \rangle$ for $a, b \in A$. Let $\Delta = dd^* + d^*d$ be the Laplace-Beltrami operator. The subspace $\mathcal{H} \subseteq A$ of harmonic elements is $\mathcal{H} = \text{Ker}(\Delta)$. There is an orthogonal decomposition $A = \text{Im}(d) \oplus \text{Im}(d^*) \oplus \mathcal{H}$, and by Hodge theory \mathcal{H}^i is canonically isomorphic to $H^i(A) = \frac{\text{Ker}(d^i)}{\text{Im}(d^{i-1})}$. Also there exists an operator $Q: A \rightarrow A$ such that $I = \Delta Q + \pi_{\mathcal{H}}$, where $I: A \rightarrow A$ is the identity map and $\pi_{\mathcal{H}}: A \rightarrow \mathcal{H}$ is the orthogonal projection onto \mathcal{H} . Moreover setting $G = d^*Q$ we get that $I = Gd + dG + \pi_{\mathcal{H}}$. We look for a perturbative solution of the equation

$$\Delta(a) + \sum_{n \geq 0, k \geq 2} O_{n,k}(a, \dots, a) \lambda^{n+k-1} = b, \quad (4)$$

where $O_{n,k}: A^{\otimes k} \rightarrow A$ is a linear operator of degree $1 - k$, $O_{n,1} = 0$ for $n \geq 1$ and $b \in A^1$. If $\mathcal{H}^1 = 0$, then $\Delta Q = I$ on A^1 and the consistency conditions of Proposition 8 hold.

Proposition 9. If $\mathcal{H}^1 = 0$, then a perturbative solution $a = \sum_{n=0}^{\infty} a_n \lambda^n$ of (4) is given by $a_n = \sum_{T \in \mathbb{T}_n} O_T(b, \dots, b)$ where $O_{(T_1, \dots, T_k)_l} = -Q(O_{l,k}(O_{T_1}, \dots, O_{T_k}))$ and $O_{\bullet} = Q$.

Next we look for a perturbative solution of an equation of the form

$$da + \sum_{n \geq 0, k \geq 2} O_{n,k}(a, \dots, a) \lambda^{n+k-1} = b, \quad (5)$$

with $a \in A^1$, $b \in A^2$, $O_{n,k}: A^{\otimes k} \rightarrow A$ operators of degree $2 - k$, $O_{n,1} = 0$ for $n \geq 1$, and $d(b) = 0$. Assume that the operators $O_{n,k}$ satisfy the generalized Leibnitz rule

$$dO_{n,k}(a_1, \dots, a_k) = \sum_{i=1}^k (-1)^{\bar{a}_1 + \dots + \bar{a}_{i-1}} O_{n,k}(a_1, \dots, d(a_i), \dots, a_k),$$

for homogeneous elements $a_1, \dots, a_k \in A$, where \bar{a} denotes the degree of an homogeneous element $a \in A$. Moreover, assume that the operator $O_{n,k}$ satisfy, for $t \geq 1$, the quadratic relations for fixed n, m, t :

$$\sum_{k+l=t+1} \sum_{1 \leq i \leq t-l+1} (-1)^{\bar{a}_1 + \dots + \bar{a}_{i-1}} O_{n,k}(a_1, \dots, O_{m,l}(a_i, \dots, a_{i+l-1}), \dots, a_t) = 0. \quad (6)$$

If $\mathcal{H}^2 \simeq H^2(A) = 0$, then $\pi_{\mathcal{H}} = 0$ and $I = Gd + dG$ on A^2 and thus G is a right inverse of d .

Proposition 10. If $\mathcal{H}^2 = 0$, then a perturbative solution of (5) is given by $a = \sum_{n=0}^{\infty} a_n \lambda^n$ where $a_n = \sum_{T \in \mathbb{T}_n} O_T(b, \dots, b)$ and $O_{(T_1, \dots, T_k)_l} = -G(O_{l,k}(O_{T_1}, \dots, O_{T_k}))$ and $O_{\bullet} = G$.

Proof. We have to show that $\sum_{T \in \mathbb{T}_n} d\tilde{O}_T(b, \dots, b) = 0$, where $\tilde{O}_{(T_1, \dots, T_k)_l} = -O_{l,k}(O_{T_1}, \dots, O_{T_k})$. Since

$$d(-O_{l,k}(O_{T_1}, \dots, O_{T_k})) = \sum_{i=1}^k \pm O_{n,k}(O_{T_1}, \dots, dG\tilde{O}_{T_i}, \dots, O_{T_k}) =$$

$$\sum_{i=1}^k \pm O_{n,k}(O_{T_1}, \dots, \tilde{O}_{T_i}, \dots, O_{T_k}) \mp O_{n,k}(O_{T_1}, \dots, Gd\tilde{O}_{T_i}, \dots, O_{T_k}).$$

Thus $\sum_{T \in T_n} d\tilde{O}_T(b, \dots, b)$ equals the sum of two terms; the first one vanishes by (6) and the second one by induction. \square

Notice the similarity between the conditions of Proposition 10 and the axioms defining A_∞ -algebras [19, 30, 31], especially when the operators $O_{n,k}$ vanish for $n > 1$. Indeed our conditions involve a countable family of operators $O_{n,k}$ satisfying a countable number of quadratic equations. It would be interesting to investigate the operadic and geometric interpretation of the conditions of Proposition 10.

4 Perturbative on-shell action

A major difficulty in the process of obtaining invariant functions by evaluating the on-shell action of classical field theories is that one can seldom find explicitly the solutions of the equations of motion. We show in this section that one can get around this problem if we evaluate instead the perturbative on-shell action. An interesting feature of the perturbative approach is that one gets automatically a hierarchy of invariants indexed by the natural numbers. The zero level is obtained by linearization of the equations of motion. Higher order invariants are obtained applying a sophisticated recursive procedure, where each step consists in solving the linear equations of motion with varying non-homogeneous term.

We are ready to discuss the main result of this work. We are going to show that under suitable conditions, made precise below, if we are given an action

$$S : F \times P \longrightarrow \mathbb{R}[[\lambda]]$$

then there are infinitely many G -invariant functions $S_{(n)} : P \longrightarrow \mathbb{R}$ with $n \geq 0$ that are constructed by evaluating the on-shell action S_{os} perturbatively. Let us then proceed to state the conditions necessary for this result. First, we assume that we have a group G which acts via a diagonal action on $F \times P$. Second, we assume that the space F of fields is provided with a non-degenerated G -invariant symmetric bilinear form $\langle \cdot, \cdot \rangle : F \otimes F \longrightarrow \mathbb{R}$. Moreover, we assume that linear operators on F can be written in the form $\langle \chi, \cdot \rangle$ for some $\chi \in F$. Expand S in powers of λ as

$$S(\varphi, p) = \sum_{n=0}^{\infty} S_n(\varphi, p) \lambda^n$$

and consider the further expansions

$$S_0(\varphi, p) = \sum_{k=1}^{\infty} \frac{Q_{0,k}(\varphi, \dots, \varphi, p)}{k} \quad \text{and for } n \geq 1 \quad \text{set} \quad S_n(\varphi, p) = \sum_{k=3}^{\infty} \frac{Q_{n,k}(\varphi, \dots, \varphi, p)}{k},$$

where $Q_{n,k} : F^{\otimes k} \times P \longrightarrow \mathbb{R}$. Notice that for $n \geq 1$ the maps $Q_{n,k}$ are defined for $k \geq 3$, this assumption fits nicely with the results of the previous section. Our third assumption is that $Q_{n,k}(g\varphi, \dots, g\varphi, gp) = Q_{n,k}(\varphi, \dots, \varphi, p)$ for each $g \in G$. By the previous assumptions we can write $Q_{n,k}(\varphi, \dots, \varphi, \psi, p) = \langle O_{n,k-1}(\varphi, \dots, \varphi, p), \psi \rangle$ where the maps $O_{n,k} : F^{\otimes k} \times P \longrightarrow F$ are such that

$$O_{n,k}(g\varphi, \dots, g\varphi, gp) = gO_{n,k}(\varphi, \dots, \varphi, p)$$

for $g \in G$. Set also $Q_{0,1}(\psi, p) = - \langle j(p), \psi \rangle$ where $j(gp) = gj(p)$ for $g \in G$. The Euler-Lagrange equations are determined by the identity

$$\frac{d}{d\epsilon} S(\varphi + \epsilon\psi)|_{\epsilon=0} = Q_{0,1}(\psi) + Q_{0,2}(\varphi, \psi) + \sum_{n \geq 0, k \geq 3} Q_{n,k}(\varphi, \dots, \varphi, \psi) \lambda^n.$$

By the previous assumptions the critical points of S are the solutions of the equation

$$O_{0,1}(\varphi, p) + \sum_{n \geq 0, k \geq 2} O_{n,k}(\varphi, \dots, \varphi, p) \lambda^n = j(p).$$

Making $\varphi \longrightarrow \lambda\varphi$, $j \longrightarrow \lambda j$, the critical points of S are determined by the equation

$$O_{0,1}(\varphi, p) + \sum_{n \geq 0, k \geq 2} O_{n,k}(\varphi, \dots, \varphi, p) \lambda^{n+k-1} = j(p). \quad (7)$$

Proposition 11. If $O_1(\cdot, p)$ is invertible for each $p \in P$, then the perturbative solution $\varphi(p) = \sum_{n=0}^{\infty} \varphi_n(p) \lambda^n$ of (7) is such that $\varphi(gp) = g\varphi(p)$ for $g \in G$.

Proof. We show that $\varphi_n(gp) = g\varphi_n(p)$ for $g \in G$. From Proposition 5 and Lemma 12 we get

$$\varphi_n(gp) = \sum_{T \in T_n} O_T(j(gp), \dots, j(gp), gp) = g \sum_{T \in T_n} O_T(j(p), \dots, j(p), p) = g\varphi_n(p).$$

□

Lemma 12. $O_T(g\alpha, \dots, g\beta, gp) = gO_T(\alpha, \dots, \beta, p)$ for $g \in G$ and $\alpha, \dots, \beta \in F$.

Proof. Assume that $T = (T_1, \dots, T_k)_l$, then

$$O_{(T_1, \dots, T_k)_l}(g\alpha, \dots, g\beta, gp) = -O_{0,1}^{-1}(O_{l,k}(O_{T_1}(g\alpha, \dots, g\kappa), \dots, O_{T_k}(g\tau, \dots, g\beta), gp), gp),$$

which by induction is equal to

$$-gO_{0,1}^{-1}(O_{l,k}(O_{T_1}(\alpha, \dots, \kappa), \dots, O_{T_k}(\tau, \dots, \beta), p), p) = gO_{(T_1, \dots, T_k)_l}(\alpha, \dots, \beta, p).$$

□

Similarly one can prove the following result.

Proposition 13. Assume that $j(p) \in O_1(V, p)$, $O_{0,1}(\cdot, p)$ has a right inverse $P(\cdot, p)$ and $\sum_{n,k \geq 2, i_1, \dots, i_k} O_{n,k}(\varphi_{i_1}, \dots, \varphi_{i_k}, p) \in O_{0,1}(V, p)$ where $n = m + \sum_{s=1}^k i_s + k - 1$. The solution $\varphi_n(p) = \sum_{T \in T_n} O_T(j, \dots, j, p)$ of (7) satisfies $\varphi_n(gp) = \varphi_n(p)$ for $g \in G$.

According to Propositions 13 if $O_{0,1}(\cdot, p)$ has a right inverse, then $\varphi(p) = \sum_{n=0}^{\infty} \varphi_n(p) \lambda^n$ given by (3) is a perturbative solution of (7). Plugging this solution in S we obtain that the perturbative on-shell action $S_{os} : P \longrightarrow \mathbb{R}[[\lambda]]$ which is given by

$$S_{os}(p) = S(\varphi(p), p) = \sum_{n=0}^{\infty} S_{(n)}(p) \lambda^n.$$

We proceed to show that the functions $S_{(n)} : P \longrightarrow \mathbb{R}$ are G -invariant. For $n \geq 0$ let R_n be the set of isomorphisms classes of labelled planar rooted trees T that can be written as $T = (T_1, \dots, T_k)_l$, where $T_s \in T_{i_s}$, $\sum_s i_s + l = n$, for $1 \leq s \leq k$ and $k \geq 1$ if $l = 0$, and $k \geq 2$ if $l \geq 1$.

Definition 14. For $T \in R_n$ let $Q_T : F^{\otimes |V_L(T)|} \longrightarrow \mathbb{R}$ be given by

$$Q_{(T_1, \dots, T_k)_l} = Q_{l,k}(O_{T_1}, \dots, O_{T_k}).$$

The proof of the following result is similar to that of Proposition 5.

Proposition 15. $S_{(n)}(p) = \sum_{T \in R_n} Q_T(j(p), \dots, j(p), p)$ for $n \geq 0$.

We are finally ready to state and prove the main result of this paper.

Theorem 16. $S_{(n)} : P \longrightarrow \mathbb{R}$ is a G -invariant function for $n \geq 0$.

Proof. If $p \in P$ and $g \in G$ then

$$S_{(n)}(gp) = \sum_{T \in R_n} Q_T(j(gp), \dots, j(gp), gp) = \sum_{T \in R_n} Q_T(j(p), \dots, j(p), p) = S_{(n)}(p).$$

□

5 Chern-Simons-Wong theory and link invariants

The relation between Chern-Simons theory and link invariants was first study in [32] and is by now a solid theory [15, 20, 27, 28], studied from a variety of points of view. A common feature of these approaches is that they work at the quantum level. It was proposed in [23] that it is possible to construct link invariants from perturbative classical non-abelian Chern-Simons action with an extra term due to Wong [4, 33]. Our desire to understand the mathematical foundations underlying the methodology of [23] was the primary motivation for this work. The

results of this section illustrate the full power of Theorem 16, which yields a hierarchy of invariant functions starting from functional actions depending equivariantly on external parameters. Let S^3 be the unit 3-sphere and \mathfrak{g} be the Lie algebra of a compact semi-simple Lie group G . Fix a symmetric non-degenerated bilinear form Tr on \mathfrak{g} invariant under the adjoint action. The space of fields

$$(\Omega^1(S^3) \otimes \mathfrak{g}) \times M(S^1, G)^n$$

consists of tuples (a, g_1, \dots, g_n) where $a \in \Omega^1(S^3) \otimes \mathfrak{g}$ is a \mathfrak{g} -valued 1-form on S^3 , and $g_i : S^1 \rightarrow G$ is a G -valued map on the unit circle. The space of parameters $E_n(S^1, S^3)$ consists of n -tuples $(\gamma_1, \dots, \gamma_n)$ such that $\gamma_i : S^1 \rightarrow S^3$ is an embedded closed curve in S^3 , and the images of the γ_i are mutually disjoint. Physically, a represents the gauge potential and g_i the chromo-electric charge of a point-particle undergoing non-abelian interactions. The trajectories in S^3 of these particles are elements of the parameter space, and we will show that the linking of these particles is tested in the process of computing the perturbative on-shell action.

Let $A(S^3)$ be the group of automorphisms of S^3 connected to the identity. $A(S^3)$ acts by pullback on $\Omega^1(S^3) \otimes \mathfrak{g}$, trivially on $M(S^1, G)^n$, and by push-forward on $E_n(S^1, S^3)$. To construct the action functional we introduce some notation. The pullback of a to S^1 via $\gamma_i : S^1 \rightarrow S^3$ is denoted by $a_i(t)$, where t is the standard coordinate on S^1 . Fix elements $c_i \in \mathfrak{g}$ and for $g_i \in M(S^1, G)$ for $1 \leq i \leq n$, let the chromo-electric charge be $c_i(t) = g_i(t)c_i g_i^{-1}(t)$, for $t \in S^1$. The covariant derivative of g_i along the i -th particle is $D_t g_i = \partial_t g_i + \lambda a_i(t) g_i$. The action functional is

$$S(a, g_1, \dots, g_n, \gamma_1, \dots, \gamma_n) = \int_{\mathbb{R}^3} Tr(a \wedge da + \frac{2}{3} \lambda a^3) + S^{int}(a, g_1, \dots, g_n, \gamma_1, \dots, \gamma_n),$$

where

$$S^{int}(a, g_1, \dots, g_n, \gamma_1, \dots, \gamma_n) = \sum_{i=1}^n \int_{\gamma_i} dt Tr(k_i g_i^{-1}(t) D_t g_i(t))$$

corresponds to the interaction of n classical Wong particles carrying non-abelian charge [4, 33]. Chern-Simons action is invariant under the group $M(S^3, G)$ of gauge transformations connected to the identity. The action of $u \in M(S^3, G)$ on $a \in \Omega^1(S^3) \otimes \mathfrak{g}$ is given by $a^u = u^{-1} a u + u^{-1} dt$. The action S^{int} is gauge invariant if we set $c_i^u = c_i$ and $g_i^u = u^{-1} g_i$. Non-abelian charges $c_i(t)$ transform in the adjoint representation $c_i(t)^u = u(t)^{-1} c_i u(t)$. With these conventions S is an $A(S^3)$ -invariant function.

The variation of S with respect to a yields the equation

$$F_a = \frac{1}{2} \sum_{i=1}^n P(\gamma_i, c_i(t)),$$

where $F_a = da + \frac{\lambda}{2}[a, a]$ is the curvature of a and $P(\gamma_i, c_i(t))$ is a Poincaré dual form defined via the identity

$$\int_{S^3} Tr(P(\gamma_i, c_i(t)) \wedge b_i) = \int_{\gamma_i} Tr(c_i(t)b_i(t))dt.$$

The variation of S with respect to g_i yields the equation $D_t c_i = \dot{c}_i + \lambda[a_i, c_i] = 0$ of conservation of non-abelian charges. Thus $c_i(t) = u_i(t) c_i u_i^{-1}(t)$, where $u_i(t) = Pexp(-\lambda \int_0^t a_i(s) ds)$ is the path ordered exponential of the gauge potential a along the curve γ_i . According to our general theory the on-shell action may be expanded as

$$S_{os}(\gamma_1, \dots, \gamma_n) = \sum_{m=0}^{\infty} S_{(m)}(\gamma_1, \dots, \gamma_n) \lambda^m$$

where each $S_{(m)}$ should be an $A(S^3)$ -invariant function of the link $(\gamma_1, \dots, \gamma_n) \in E_n(S^1, S^3)$. Using $g_i(t) = u_i(t)g_i(0)$ one can check that $S_{os}^{int}(\gamma_1, \dots, \gamma_n) = 0$. Let $\hat{\cdot}: \mathfrak{g} \longrightarrow End(\mathfrak{g})$ be the adjoint representation of \mathfrak{g} given by $\hat{x}(y) = [x, y]$ for $x, y \in \mathfrak{g}$. From the equation $D_t c_i = \dot{c}_i + \lambda \hat{a}_i(c_i(t)) = 0$ we see that $c_i(t) = Pexp(-\lambda \int_0^t \hat{a}_i dt) c_i$, and thus the equation $F_a = \frac{1}{2} \sum_{i=1}^n P(\gamma_i, c_i(t))$ becomes

$$da = -\frac{\lambda}{2}[a, a] + \frac{1}{2} \sum_{i=1}^n P(\gamma_i) c_i + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{i=1}^n P(\gamma_i, c_{m,i}(t)) c_i \lambda^m,$$

where

$$c_{m,i}(t) = \int_{\Delta_{0,t}^m} \bigwedge_{j=1}^m e_{i,j}^*(\hat{a}), \quad \Delta_{0,t}^m = \{(x_1, x_2, \dots, x_m) \mid 0 \leq x_j \leq t \text{ and } x_j \leq x_k \text{ if } j \leq k\},$$

and the map $e_{i,j}: \Delta_{0,t}^m \longrightarrow S^3$ is given by $e_{i,j}(x_1, x_2, \dots, x_m) = \gamma_i(x_j)$.

We look for a perturbative solution $a = \sum_p a_{(p)} \lambda^p$ of the equation of motion. The corresponding recursive system of linear equations is given by $da_{(0)} = \frac{1}{2} \sum_{i=1}^n P(\gamma_i) c_i$, and for $p \geq 1$

$$da_{(p)} = -\frac{1}{2} \sum_{s_1+s_2=p-1} [a_{(s_1)}, a_{(s_2)}] + \frac{1}{2} \sum_{m=1}^p \sum_{i=1}^n P(\gamma_i, c_{m,i}(t)) c_i,$$

where

$$c_{m,i}(t) = \sum_{s_1+\dots+s_m=p-m} \int_{\Delta_{0,t}^m} \bigwedge_{j=1}^m e_{i,j}^*(\hat{a}_{(s_j)}).$$

Similarly the perturbative on-shell action is $S_{os} = \sum_{m=0}^{\infty} S_{(m)} \lambda^m$ where for $m \geq 0$ we have

$$S_{(m)} = \int_{S^3} \sum_{s_1+s_2=p} Tr(a_{(s_1)} da_{(s_2)}) + \frac{2}{3} \int_{S^3} \sum_{s_1+s_2+s_3=p-1} Tr(a_{(s_1)} a_{(s_2)} a_{(s_3)}).$$

Thus $S_{(0)}$ is given by $S_{(0)} = \int_{S^3} \text{Tr}(a_{(0)} da_{(0)})$. If $\Sigma_i : D^1 \longrightarrow M$ is such that $\partial(\Sigma_i) = \gamma_i$, i.e. Σ_i a Seifert surface for γ_i , then we have that $a_{(0)} = \frac{1}{2} \sum_{i=1}^n P(\Sigma_i) c_i$ and we get

$$S_{(0)} = \frac{1}{4} \sum_{i,j=1} \text{Tr}(c_i c_j) \int_{S^3} P(\Sigma_i) P(\gamma_j) = \frac{1}{4} \sum_{i,j=1} \text{Tr}(c_i c_j) \text{lk}(\gamma_i, \gamma_j).$$

$S_{(0)}$ is a linear combination of linking numbers, so it is a link invariant as predicted from our general theory. We proceed to compute explicitly $S_{(1)}$ which is given by

$$S_{(1)} = \int_{S^3} \text{Tr}(2a_{(0)} da_{(1)} + \frac{2}{3} a_{(0)}^3).$$

We know that $a_{(0)} = \frac{1}{2} \sum_{i=1}^n P(\Sigma_i) c_i$ and

$$da_{(1)} = -\frac{1}{2} [a_{(0)}, a_{(0)}] + \frac{1}{2} \sum_{i=1}^n P(\gamma_i, c_{1,i}(t)) c_i,$$

where $c_{1,i}(t) = \frac{1}{2} \sum_{j=1}^n \int_{\Delta_{0,t}^1} e_{1,j}^* P(\Sigma_i) \widehat{c}_i$. Plugging these identities in the previous expression for $S_{(1)}$ we obtain

$$S_{(1)} = -\frac{1}{4} \sum_{i,j,k} \text{Tr}(c_i [c_j, c_k]) \left(\frac{1}{3} \int_{S^3} P(\Sigma_i) P(\Sigma_j) P(\Sigma_k) + \int_{\Delta_{0,1}^2} e_{1,j}^* P(\Sigma_k) \wedge e_{2,j}^* P(\Sigma_i) \right).$$

The first summand in the formula above should be clear. The second summand arises from

$$\frac{1}{4} \sum_{i,j,k} \int_{S^3} \text{Tr}(P(\Sigma_i) c_i P(\gamma_j, \int_{\Delta_{0,t}^1} e_{1,j}^* (P(\Sigma_k) \widehat{c}_k) c_j),$$

or equivalently

$$\frac{1}{4} \sum_{i,j,k} \text{Tr}(c_i [c_k, c_j]) \int_{S^3} \text{Tr}(P(\Sigma_i)) P(\gamma_j, \int_{\Delta_{0,t}^1} e_{1,j}^* (P(\Sigma_k))).$$

By the defining properties of Poincaré forms and antisymmetry of the Lie bracket, the later expression is equal to

$$-\frac{1}{4} \sum_{i,j,k} \text{Tr}(c_i [c_j, c_k]) \int_{\Delta_{0,1}^2} e_{1,j}^* P(\Sigma_k) \wedge e_{2,j}^* P(\Sigma_i).$$

The formula obtained for $S_{(1)}$ is a link invariant with a crystal clear geometric meaning: the first summand counts triple intersections of the corresponding Seifert surfaces, the second summand counts pairs of points s, t in the parametrization of loop γ_i , such that $\gamma_i(s) \in \Sigma_k$ and $\gamma_i(t) \in \Sigma_j$. In the computation of $S_{(1)}$ we make use of the identity

$$da_{(1)} = -[a_{(0)}, a_{(0)}] + \sum_{i=1}^n P(\gamma_i, c_{1,i}(t)) c_i,$$

thus we assumed that the right hand side of this identity is a closed two-form. This assumption is by no means trivial and does not hold universally. Indeed it imposes a severe restriction on the type of links for which the invariant $S_{(1)}$ is well-defined: the linking number of each pair of loops in the link must vanish. The Borromean rings is an example of link for which the invariant $S_{(1)}$ is well-defined and non-vanishing. For a proof of this and other interesting facts regarding the invariant $S_{(1)}$ the reader may consult [23]. The reader should notice that $S_{(1)}$ is the second Milnor's invariant [25] for links in S^3 , and thus our method provides an interpretation for that invariant coming from perturbative Lagrangian physics. We expect that the higher order invariants $S_{(n)}$ correspond with higher order Milnor's invariants which can be computed using higher order Massey products [26].

6 Yang-Mills theory and area invariants

In this section we show that it is possible to obtain invariants of configurations of immersed curves in the plane from Yang-Mills-Wong action. To our knowledge results of this type have seldom been reported – unlike the relation between links and Chern-Simons theory – perhaps because the space of immersed curves in the plane, considered up to area preserving diffeomorphisms, has not been deeply studied in the mathematical literature. The example consider in this section is studied in full details in [13], here we only highlight the results of that paper that are useful to illustrate yet another application of our method.

The basic settings is quite similar to those for Chern-Simons-Wong action. The space of fields is $(\Omega^1(\mathbb{R}^2) \otimes \mathfrak{g}) \times M(S^1, G)^n$. The space of parameters $I_n(S^1, \mathbb{R}^2)$ consists of n -tuples $(\gamma_1, \dots, \gamma_n)$ such that $\gamma_i : S^1 \rightarrow \mathbb{R}^2$ is an immersed closed curve in \mathbb{R}^2 , and the images of the γ_i intersect, if they do, in transversal double points. The group of symmetries for the Yang-Mills-Wong action is the group of area preserving diffeomorphisms of \mathbb{R}^2 . As before we fix $c_i \in \mathfrak{g}$ and for $1 \leq i \leq n$ we let $g_i \in M(S^1, G)$. The action functional is given by

$$S(a, g_1, \dots, g_n, \gamma_1, \dots, \gamma_n) = \int_{\mathbb{R}^2} Tr(F_a \wedge *F_a) + \sum_{i=1}^n \int_{\gamma_i} d\tau Tr(k_i g_i^{-1}(\tau) D_\tau g_i(\tau)),$$

where $F_a = da + \frac{\lambda}{2}[a, a]$ and $*$ is the Hodge star operator. According to our general theory the on-shell action may be expanded as

$$S_{os}(\gamma_1, \dots, \gamma_n) = \sum_{m=0}^{\infty} S_{(m)}(\gamma_1, \dots, \gamma_n) \lambda^m$$

where each $S_{(m)}$ should be a function of $(\gamma_1, \dots, \gamma_n) \in I_n(S^1, \mathbb{R}^2)$ invariant under area preserving diffeomorphisms of \mathbb{R}^2 . One can compute $S_{(0)}$ and $S_{(1)}$ and check that they are indeed invariants

under area preserving diffeomorphisms. The invariant $S_{(0)}$ admits the fairly simple expression

$$S_{(0)} = \sum_{i,j} \text{Tr}(c_i c_j) J(\gamma_i, \gamma_j),$$

where the functions $J(\gamma_i, \gamma_j)$ have the following geometric interpretation. A generic immersed curve in \mathbb{R}^2 induces a partition of \mathbb{R}^2 into a finite number of compact blocks and an unbounded block. The function $J(\gamma_i, \gamma_j)$ is the sum of the signed areas of the intersections of the finite blocks of γ_i with the finite blocks of γ_j . In complete analogy with the Chern-Simons-Wong case, the geometric interpretation of $S_{(1)}$ takes into account not just the areas of the intersection blocks, but also the order in which the intersection blocks appear for several, at least three, curves. Again $S_{(1)}$ is only well-defined for an appropriated choice of curves. In [13] we describe explicitly three planar curves – a planar version of the Borromean rings – for which $S_{(1)}$ is well-defined and non-vanishing.

7 Final remarks

We introduced a method that yields invariant functions from classical field theories. In the perturbative regime we actually obtain a countable hierarchy of invariants. Our construction leaves many open problems and suggest new lines of research. For Chern-Simons-Wong action and 2-dimensional Yang-Mills-Wong action we are, at this point, only able to compute the first two invariants of the hierarchy. Though Theorem 16 provides explicit formulae for the higher order invariants, and our computations suggest that the consistency equations are satisfied, a rigorous proof is needed. We expect the higher order link invariants arising in the computation of the perturbative Chern-Simons-Wong on-shell action, to be closely related to Milnor's link invariants [25].

We believe our methods can be usefully applied to other classical field theories. In particular, it may be rewarding to look at Yang-Mills-Wong action in higher dimensions, it should yield conformal invariants associated with closed curves in spacetime. It may also be interesting to apply our methods to the generalized Chern-Simons action of [29], it should yield invariants related with Chas-Sullivan product in string topology [9, 10]. In our study of perturbative solutions we saw that the invertibility of the quadratic part of the action plays a fundamental role. Often the quadratic part is not invertible and new techniques are required in order to get invariants. One possibility is to introduce, as in the quantum case, fermionic variables and replace the action with a new one with invertible quadratic part. Thus, it is plausible that in the classical perturbative regime, the BRST and BV procedures may still play a role. Another possibility arises when the inverse of the quadratic part of the action is not quite well-defined, but rather a singular operator. In this case techniques from renormalization [11] may become useful in order

to replace invariants given by ill-defined divergent integrals, by their renormalized values. In recent years it has become clear that many constructions in field theory [2, 12, 14], as well as in other branches of physics and mathematics [3, 5, 6, 7, 21] admit categorical analogues. It would be interesting to investigate the categorical foundations of the method introduced in this paper.

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